

# Maximin equilibrium

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**Maximin equilibrium**

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# Maximin equilibrium\*

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## Abstract

We introduce a new concept which extends von Neumann and Morgenstern's maximin strategy solution by incorporating 'individual rationality' of the players. Maximin equilibrium, extending Nash's value approach, is based on the evaluation of the strategic uncertainty of the whole game. We show that maximin equilibrium is invariant under strictly increasing transformations of the payoffs. Notably, every finite game possesses a maximin equilibrium in pure strategies. Considering the games in von Neumann-Morgenstern mixed extension, we demonstrate that the maximin equilibrium value is precisely the maximin (minimax) value and it coincides with the maximin strategies in two-person zerosum games. We also show that for every Nash equilibrium that is not a maximin equilibrium there exists a maximin equilibrium that Pareto dominates it. Hence, a strong Nash equilibrium is always a maximin equilibrium. In addition, a maximin equilibrium is never Pareto dominated by a Nash equilibrium. Finally, we discuss maximin equilibrium predictions in several games including the traveler's dilemma.

*JEL-Classification:* C72

*Keywords:* Non-cooperative games, maximin strategy, zerosum games.

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# 1 Introduction

In their ground-breaking book, von Neumann and Morgenstern (1944, p. 555) describe the maximin strategy<sup>1</sup> solution for two-person games as follows:

“There exists precisely one solution. It consists of all those imputations where each player gets individually at least that amount which he can secure for himself, while the two get together precisely the maximum amount which they can secure together. Here the ‘amount which a player can get for himself’ must be understood to be the amount which he can get for himself, irrespective of what his opponent does, even assuming that his opponent is guided by the desire to inflict a loss rather than to achieve a gain.”

This immediately gives rise to the following question: “What happens when a player acts according to the maximin principle but knowing that other players do not necessarily act in order to decrease his utility?”. We are going to capture this type of behavior by assuming that players are ‘individually rational’ and letting this be common knowledge among players. In other words, the contribution of the current paper can be considered as incorporating the maximin principle and ‘rationality’ of the players in one concept calling it maximin equilibrium. Our solution coincides with maximin strategy solution when the rationality assumption is dropped.

Note that it is recognized and explicitly stated by von Neumann and Morgenstern several times that their approach can be questioned by not capturing the cooperative side of non-zerosum games. But this did not seem to be a big problem at that time and it is stated that the applications of the theory should be seen in order to reach a conclusion.<sup>2</sup> After more than a half-century of research in this area, maximin strategies are indeed considered to be too defensive in non-strictly competitive games in the literature. Since a maximin strategist plays any game as if it is a zerosum game, this leads to an ignorance of her opponent’s utilities and hence the preferences of her opponent. These arguments call for a revision of the maximin strategy concept in non-zerosum games.

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<sup>1</sup>We would like to note that the famous minimax (or maximin) theorem was proved by von Neumann (1928). Therefore, it is generally referred as von Neumann’s theory of games in the literature.

<sup>2</sup>For example, see von Neumann and Morgenstern (1944, p. 540).

In Section 2, we present the framework and introduce the concept of maximin equilibrium. Maximin equilibrium extends Nash’s value approach to the whole game and evaluates the strategic uncertainty of the game by following a similar method as von Neumann’s maximin strategy notion. We show that every finite game possesses a maximin equilibrium in pure strategies. Moreover, maximin equilibrium is invariant under strictly increasing transformations of the utility functions of the players. In Section 3, we extend the analysis to the games in von Neumann-Morgenstern mixed extension. We demonstrate that maximin equilibrium exists in mixed strategies too. We also show that for every Nash equilibrium that is not a maximin equilibrium there exists a maximin equilibrium that Pareto dominates it. Hence, a strong Nash equilibrium is always a maximin equilibrium. In addition, a maximin equilibrium is never Pareto dominated by a Nash equilibrium. Furthermore, we show by examples that maximin equilibrium is neither a coarsening nor a special case of correlated equilibrium or rationalizable strategy profiles.

In Section 4, we show that a strategy profile is a maximin equilibrium if and only if it is a pair of maximin strategies in two-person zerosum games. In particular, the maximin equilibrium value is precisely the minimax value whenever the latter exists. In Section 5, we discuss the maximin equilibrium in  $n$ -person games. All the results provided in Section 2 and in Section 3 hold in  $n$ -person games.

## 2 Maximin equilibrium

In this paper, we use a framework for the analysis of interactive decision making environments as described by von Neumann and Morgenstern (1944, p. 11):

One would be mistaken to believe that it [the uncertainty] can be obviated, like the difficulty in the Crusoe case mentioned in footnote 2 on p. 10, by a mere recourse to the devices of the theory of probability. Every participant can determine the variables which describe his own actions but not those of the others. Nevertheless those ‘alien’ variables cannot, from his point of view, be described by statistical assumptions. This is because the others are guided, just as he himself, by rational principles –whatever that may mean– and no *modus procedendi* can be correct which

does not attempt to understand those principles and the interactions of the conflicting interests of all participants.

For simplicity, we assume that there are two players whose finite sets of pure actions are  $X_1$  and  $X_2$  respectively. Moreover, players' preferences over the outcomes are assumed to be a weak order (i.e. transitive and complete) so that we can represent those preferences by the ordinal utility functions  $u_1, u_2 : X_1 \times X_2 \rightarrow \mathbb{R}$  which depends on both players' actions. As usual, the notation  $x$  in  $X = X_1 \times X_2$  represents a strategy profile.<sup>3</sup> In short, a two-person non-cooperative game  $\Gamma$  can be denoted by the tuple  $(\{1, 2\}, X_1, X_2, u_1, u_2)$ . We distinguish between the game  $\Gamma$  and its von Neumann-Morgenstern mixed extension. Clearly, the mixed extension of a game requires more assumptions to be made and it will be treated separately in Section 3. When it is not clear from the context, we refer the original game as the pure game or the deterministic game to not to cause a confusion with the games in mixed extension. Starting from simple strategic decision making situations, we firstly introduce a deterministic theory of games in this section.<sup>4</sup>

As it is formulated and explained by von Neumann and Morgenstern (1944), playing a game is basically facing an uncertainty which can not be resolved by statistical assumptions. This is actually the crucial difference between strategic games and decision problems. Our aim is to extend von Neumann's approach on resolving this uncertainty.

Suppose that Alfa (he) and Beta (she) make a non-binding agreement  $(x_1, x_2)$  in  $X$  in a two-person game. Alfa faces an uncertainty by keeping the agreement since he does not know whether Beta will keep it. Von Neumann's maximin method to evaluate this uncertainty is to calculate the minimum payoff of Alfa with respect to all conceivable deviations by Beta.<sup>5</sup> That is, Alfa's evaluation  $v_{x_1 x_2}$  (or the utility) of keeping the agreement  $(x_1, x_2)$  is  $v_{x_1 x_2} = \min_{x'_2 \in X_2} u_1(x_1, x'_2)$ . Note that for all  $x_2$ , the evaluation of Alfa for the profile  $(x_1, x_2)$  is the same, i.e.  $v_{x_1 x_2} = v_{x_1 x'_2}$  for all  $x'_2 \in X_2$ . Therefore, it is possible to attach a unique evaluation  $v_{x'_1}$  for every strategy  $x'_1 \in X_1$  of Alfa. Second step is to make a comparison between those evaluations

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<sup>3</sup>As is standard in game theory, we assume that what matters is the consequence of strategies (consequentialist approach) so that we can define the utility functions over the strategy profiles.

<sup>4</sup>Note that all the definitions we present can be extended in a straightforward way to  $n$ -person games which will be introduced in Section 5.

<sup>5</sup>Because, it is assumed that Beta might have a desire to inflict a loss for Alfa. Note that von Neumann also included mixed strategies but here we would like to keep it simple.

of the strategies. For that, von Neumann takes the maximum of all such evaluations  $v_{x'_1}$  with respect to  $x'_1$  which yields a unique evaluation for the whole game, i.e. the value of the game is  $v_1 = \max_{x'_1 \in X_1} v_{x'_1}$ . In other words, the unique utility that Alfa can guarantee by facing the uncertainty of playing this game is  $v_1$ . Accordingly, it is recommended that Alfa should choose a strategy  $x_1^* \in \arg \max_{x'_1 \in X_1} v_{x'_1}$  which guarantees the value  $v_1$ .

We would like to extend von Neumann's method in such a way that Alfa takes into account the 'individual rationality' of Beta when making the evaluations and vice versa. Let us fix some terminology. As usual, a strategy  $x'_i \in X_i$  is said to be a *profitable deviation* for player  $i$  with respect to the profile  $(x_i, x_j)$  if  $u_i(x'_i, x_j) > u_i(x_i, x_j)$ .

**Definition 1.** A player is called *individually rational at  $x$*  in  $X$  if she *does not* make a non-profitable deviation from it.

We assume that players are individually rational, each player assumes that the other players are individually rational and that this is common knowledge.<sup>6</sup> Let us construct the approach we take step by step and state its implications. We have proposed a notion of individual rationality which allows Beta to keep her agreement or to deviate to a strategy for which she has strict incentives to do so. This is reminiscent of individual rationality constraint in economics in the sense that individually rational behavior always yield higher or equal utility than individually non-rational behavior. By this assumption, Alfa can rule out non-profitable deviations of Beta from the agreement  $(x_1, x_2)$  which helps decreasing the level of uncertainty he is facing. Now, Alfa's evaluation  $v_1(x_1, x_2)$  of the uncertainty for keeping the agreement  $(x_1, x_2)$  can be defined as the minimum utility he would receive under any individually rational behavior of Beta. Let us define the value function formally.

**Definition 2.** Let  $\Gamma = (X_1, X_2, u_1, u_2)$  be a two-person game. The function  $v : X \rightarrow \mathbb{R} \times \mathbb{R}$  is called the *value function* of  $\Gamma$  if for every  $i \neq j$  and for all  $x = (x_i, x_j) \in X$ , the  $i$ 'th component of  $v = (v_i, v_j)$  satisfies

$$v_i(x) = \min \left\{ \inf_{x'_j \in B_j(x)} u_i(x_i, x'_j), u_i(x) \right\},$$

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<sup>6</sup>See Lewis (1969) for a detailed discussion and see Aumann (1976) for a formal definition of common knowledge in a Bayesian setting.

where the better response correspondence of player  $j$  with respect to  $x$  is defined as

$$B_j(x) = \{x'_j \in X_j | u_j(x_i, x'_j) > u_j(x)\}.$$

*Remark.* Note that for all  $x$  and all  $i$ , we have  $u_i(x) \geq v_i(x)$ . This is because one cannot increase a payoff but can only (weakly) decrease it, by definition of the value function.

As a consequence, it is not in general true for a strategy  $x'_2 \neq x_2$  that we have the equality  $v_1(x_1, x_2) = v_1(x_1, x'_2)$ . Because, the better response set of Beta with respect to  $(x_1, x_2)$  is not necessarily the same as the better response set of her with respect to  $(x_1, x'_2)$ . Therefore, we cannot assign a unique value to every strategy of Alfa anymore. Instead, the evaluation of the uncertainty can be encoded in the strategy profile as in the value notion of Nash (1950, 1951). Nash defines the value of the game (henceforth the Nash-value) to a player as the payoff that the player receives from a Nash equilibrium when all the Nash equilibria lead to the same payoff for the player. We extend Nash's value approach to the full domain of the game, that is, we assign a value to each single strategy profile including, of course, the Nash equilibria. Notice that when a strategy profile is a Nash equilibrium, the value of a player at this profile is precisely her Nash equilibrium payoff.<sup>7</sup> In particular, if the Nash-value exists for a player then the player's value of every Nash equilibria is the Nash-value of that player. As a result of assigning a value to the profiles rather than the strategies, we can no longer refer to a strategy in the same spirit of a maximin strategy since a strategy in this setting only makes sense as a part of a strategy profile as in a Nash equilibrium. But note that there are two evaluations that are attached to the profile  $(x_1, x_2)$ , one for Alfa and one for Beta since she also is doing similar inferences as him.

To illustrate what a value function of a game looks like, let us consider the game  $\Gamma$  in Figure 1 which is played by Alfa and Beta. It can be interpreted as the prisoner's dilemma game with an option to remain silent. Each prisoner has three options to choose from, namely remain 'Silent', 'Deny' or 'Confess' and let the utilities be as in Figure 1. Notice that if the strategy 'Silent' is removed from the game for both players then we would obtain the prisoner's dilemma.

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<sup>7</sup>This is because there is no individually rational deviation from a Nash equilibrium, hence infimum over empty set is plus infinity which implies the value of a player at a Nash equilibrium equals its payoff.



		Silent	Deny	Confess
$\Gamma =$	Silent	100, 100	100, 105	0, 1
	Deny	105, 100	95, 95	0, 200
	Confess	1, 0	200, 0	1, 1

		Silent	Deny	Confess
$v(\Gamma) =$	Silent	100, 100	100, 0	0, 1
	Deny	0, 100	0, 0	0, 1
	Confess	1, 0	1, 0	1, 1

Figure 1: Prisoner's dilemma with an option to remain silent and its value function.

Suppose that the prisoners Alfa and Beta are in the same cell and they can freely discuss what to choose before they submit their strategies. However, they will make their choices in separate cells, that is, non-binding pre-game communication is allowed. Suppose that Beta is trying to convince Alfa to make an agreement on playing, for example, the profile (Deny, Deny). Alfa would fear that Beta may not keep her agreement and may unilaterally deviate to 'Confess' leaving him a utility of 0. Accordingly, the value of the profile (Deny, Deny) to Alfa is 0 as shown in the bottom table in Figure 1. Now, suppose somebody offers to make an agreement on (Silent, Silent). Beta would not fear a unilateral profitable deviation 'Deny' of Alfa since she still gets 100 in that case. Alfa's utility does not change too in case of a unilateral profitable deviation of Beta to 'Deny'. In other words, the value of the profile (Silent, Silent) is (100, 100) which is equal to its payoff vector in  $\Gamma$ .

The second and the last step is to make comparisons between the evaluations of the strategy profiles. We maximize the value function by the Pareto optimality principle. Now, let us formally define the maximin equilibrium.

**Definition 3.** Let  $(X_1, X_2, u_1, u_2)$  be a two-person game and let  $v = (v_i, v_j)$  be the value function of the game. A strategy profile  $x = (x_i, x_j)$  where  $i \neq j$  is called *maximin equilibrium* if for every player  $i$  and every  $x' \in X$ ,  $v_i(x') > v_i(x)$  implies  $v_j(x') < v_j(x)$ .

Notice that if we do not assume individual rationality of the players then we recover maximin strategy concept. That is, our solution would coincide with maximin strategy solution. To see this, we may interpret the better

response correspondence of player  $j$  with respect to a profile  $x$ , i.e.  $B_j(x)$ , as being the belief of player  $i$  about player  $j$ 's possible strategies. Maximin strategy corresponds to the case in which a player's belief about her opponent is the whole strategy set of the opponent. That is, player  $i$  does not take individual rationality of the opponent into account. With this interpretation, the maximin principle can be incorporated with stronger or weaker rationality assumptions, even with different ones for different players, by following the same method we follow in this section. *Mutatis mutandis*, there would not be a change in the results of this section.

Going back to the example in Figure 1, observe that the game has a unique Nash equilibrium (Confess, Confess) with a payoff vector of  $(1, 1)$ . Observe also that the profile (Silent, Silent) is the Pareto dominant profile of the value function, so it is the maximin equilibrium with a value of  $(100, 100)$ . Moreover, the maximin equilibrium (Silent, Silent) has another property which may deserve attention. Suppose that players agree on playing it. Alfa has a chance to make a unilateral profitable deviation to 'Deny' but he cannot rule out a potential profitable deviation of Beta to the strategy 'Deny'. If this happens, Alfa would receive 95 which is strictly less than what he would receive if he did not deviate to 'Deny'. But Beta is also in the exactly same situation. As a result, it seems that none of them would actually deviate from the agreement (Silent, Silent).

We obtain maximin equilibrium by evaluating each single strategy profile in a game. One of the reasons of extending Nash (1950)'s value argument is the following. A Nash equilibrium is solely based on the evaluation of the outcomes that might occur as a consequence of a player choosing one strategy with the outcomes that might occur as a consequence of an opponent choosing another strategy. Therefore, it seems to be quite questionable whether the Nash-value represents an evaluation of the strategic uncertainty of the *whole* game or only of these outcomes. Since a Nash equilibrium completely ignores the outcomes that might occur under any other strategy choices of the players no matter how high their utilities are, this ignorance might lead to a disastrous outcome for both players in strategic games. One can see this clearly in the traveler's dilemma game which is illustrated in Figure 2 and which was introduced by Basu (1994). If players play the unique Nash equilibrium, then they ignore a large part of the game which is mutually beneficial for both of them, but mutually beneficial trade is perhaps one of the most basic principles in economics.

	100	99	...	3	2
100	100, 100	97, 101	...	1, 5	0, 4
99	101, 97	99, 99	...	1, 5	0, 4
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
3	5, 1	5, 1	...	3, 3	0, 4
2	4, 0	4, 0	...	4, 0	2, 2

Figure 2: Traveler's dilemma

In the traveler's dilemma, the payoff function of a player  $i$  if she plays  $x_i$  and her opponent plays  $x_j$  is defined as  $u_i(x_i, x_j) = \min\{x_i, x_j\} + r \cdot \text{sgn}(x_j - x_i)$  for all  $x_i, x_j$  in  $X = \{2, 3, \dots, 100\}$  where  $r > 1$  determines the magnitude of reward and punishment which is 2 in the original game. Regardless of the magnitude of the reward/punishment, the unique Nash equilibrium is (2, 2) which is also the unique outcome of the process of iterated elimination of strictly dominated strategies.

It is shown by many experiments that players do not on average choose the Nash equilibrium strategy and that changing the reward/punishment parameter  $r$  affects the behavior observed in experiments. Goeree and Holt (2001) found that when the reward is high, 80% of the subjects choose the Nash equilibrium strategy but when the reward is small about the same percent of the subjects choose the highest. This finding is a confirmation of Capra et al. (1999). There, play converged towards the Nash equilibrium over time when the reward was high but converged towards the other extreme when the reward was small. On the other hand, Rubinstein (2007) found (in a web-based experiment without payments) that 55% of 2985 subjects choose the highest amount and only 13% choose the Nash equilibrium where the reward was small. These results are actually not unexpected. The irony is that if both players choose almost<sup>8</sup> any 'irrational' strategy but their Nash equilibrium strategy, then they both get strictly more payoff than they would get by playing the Nash equilibrium. Moreover, the strategy '2' is the worst reply in all those cases. In fact, the Nash equilibrium is the only profile which has this property in the game.

To find the maximin equilibria we first need to compute the value of the traveler's dilemma. The value function of player  $i$  is given by

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<sup>8</sup>If one modifies the payoffs of the game such that  $u_i(x_i, 3) = 2.1$  and  $u_i(x_i, 4) = 2.1$  for all  $i$  and all  $x_i \in \{4, 5, \dots, 100\}$ , then one can even remove 'almost' from this sentence.

$$v_i(x_i, x_j) = \begin{cases} x_j - 2, & \text{if } x_i > x_j \text{ for } x_i \in X \\ x_i - 3, & \text{if } x_i = x_j \text{ for } x_i \in X \setminus \{2\} \\ 2, & \text{if } x_i = x_j = 2 \\ x_i - 5, & \text{if } x_i < x_j \text{ for } x_i \in X \setminus \{4, 3, 2\} \\ 0, & \text{if } x_i < x_j \text{ for } x_i \in \{4, 3, 2\}. \end{cases}$$

Observe that the maximum of the value function is (97, 97) which is assumed at (100, 100). Hence, the profile (100, 100) is the unique maximin equilibrium and (97, 97) is the value of it. Note that as the reward parameter  $r$  increases, the value of the maximin equilibrium decreases. When  $r$  is higher than or equal to 50, the unique maximin equilibrium becomes the profile (2, 2) which is also the unique Nash equilibrium of the game. This seems to explain both the convergence of play to (100, 100) when the reward is small, and the convergence of play to (2, 2) when the reward is big.

An ordinal utility function is unique up to strictly increasing transformations. Therefore, it is crucial for a solution concept (which is defined with respect to ordinal utilities) to be invariant under those operations. The following proposition shows that maximin equilibrium possesses this property.

**Proposition 1.** *Maximin equilibrium is invariant under strictly increasing transformations of the utility function of the players.*

*Proof.* Let  $\Gamma = (X_i, X_j, u_i, u_j)$  and  $\hat{\Gamma} = (X_i, X_j, \hat{u}_i, \hat{u}_j)$  be two games such that  $\hat{u}_i$  and  $\hat{u}_j$  are strictly increasing transformations of  $u_i$  and  $u_j$  respectively. Firstly, we show that the components  $\hat{v}_i$  and  $\hat{v}_j$  of the value function  $\hat{v}$  are strictly increasing transformations of the components  $v_i$  and  $v_j$  of  $v$ , respectively. Notice that  $B_j(x) = \hat{B}_j(x)$ , that is

$$\{x'_j \in X_j | u_j(x_i, x'_j) > u_j(x)\} = \{x'_j \in X_j | \hat{u}_j(x_i, x'_j) > \hat{u}_j(x)\}.$$

It implies that  $\arg \min_{x'_j \in B_j(x)} u_i(x_i, x'_j) = \arg \min_{x'_j \in \hat{B}_j(x)} \hat{u}_i(x_i, x'_j)$  such that  $v_i(x) = \min\{u_i(x_i, \bar{x}_j), u_i(x)\}$  and  $\hat{v}_i(x) = \min\{\hat{u}_i(x_i, \bar{x}_j), \hat{u}_i(x)\}$  for some  $\bar{x}_j \in \arg \min_{x'_j \in B_j(x)} u_i(x_i, x'_j)$ . Since  $\hat{u}_i$  is a strictly increasing transformation of  $u_i$ , we have either  $v_i(x) = u_i(x_i, \bar{x}_j)$  if and only if  $\hat{v}_i(x) = \hat{u}_i(x_i, \bar{x}_j)$  or  $v_i(x) = u_i(x)$  if and only if  $\hat{v}_i(x) = \hat{u}_i(x)$  for all  $x_i, x_j$  and all  $\bar{x}_j$ . It follows that showing  $v_i(x) \geq v_i(x')$  if and only if  $\hat{v}_i(x) \geq \hat{v}_i(x')$  is equivalent to showing  $u_i(x) \geq u_i(x')$  if and only if  $\hat{u}_i(x) \geq \hat{u}_i(x')$  for all  $x, x'$  in  $X$  which is correct by our supposition.

	a	b	c
a	1, 1	3, 3	1, 1
b	3, 1	3, 3	3, 4
c	3, 3	1, 3	4, 1

	a	b	c	I
a	1, 1	3, 3	1, 1	-1, 0
b	3, 1	3, 3	3, 4	-1, 0
c	3, 3	1, 3	4, 1	-1, 0
I	0, -1	0, -1	0, -1	0, 0

Figure 3: Two games  $\Gamma$  (left) and  $\Gamma'$  (right). In the former, the payoffs to the Nash equilibria and to the maximin strategies are the same while it changes in the latter.

Secondly, a profile  $y$  is a Pareto optimal profile with respect to  $v$  if and only if it is Pareto optimal with respect to  $\hat{v}$  because each  $v_i$  is a strictly increasing transformation of  $\hat{v}_i$ . As a result, the set of maximin equilibria of  $\Gamma$  and  $\hat{\Gamma}$  are the same.  $\square$

The following proposition shows the existence of maximin equilibrium in pure strategies. This may be especially a desired property in games where players cannot or are not able to use a randomization device. It might be also the case that a commitment of a player to a randomization device is implausible. In those games, we can make sure that there exists at least one maximin equilibrium.

**Theorem 1.** *Every finite game has a maximin equilibrium in pure strategies.*

*Proof.* Since Pareto dominance relation is reflexive and transitive a Pareto optimal strategy profile with respect to the value function of a finite game always exists.  $\square$

Moreover, maximin equilibria are invariant under addition of “irrelevant strategies” to a game. In other words, suppose that we add new strategies to a game  $\Gamma$  calling the new game  $\Gamma'$  and that all new outcomes are strictly less preferred to the outcomes in  $\Gamma$ . Then the set of maximin equilibria in  $\Gamma'$  are the same as the ones in  $\Gamma$ . For example, let us consider the games shown in Figure 3. All the Nash equilibria yield the same (expected) payoff vector (3, 3) in  $\Gamma$ . Observe that the unique maximin strategy is ‘b’ for both players which guarantees each of them to receive a payoff of 3. Notice also that (b, b) is the only maximin equilibrium which is not a Nash equilibrium in this game.

Although the point we want to make is different, it is of importance to note the historical discussion about this type of games where the Nash equilibria payoffs are equal to the payoffs that can be guaranteed by playing maximin strategies. Harsanyi (1966) postulates that players should use their maximin strategies in those games which he calls unprofitable. Luce and Raiffa (1957) and Aumann and Maschler (1972) argue that maximin strategies seem preferable in those games. In short, in the games similar to Figure 3, the arguments supporting maximin strategies are so strong that it led some game theory giants to prefer them over the Nash equilibria of the game. These arguments, however, may disappear when we add an ‘irrelevant’ strategy ‘I’ to the game for both players. Notice that the Nash equilibria in  $\Gamma$  are also Nash equilibria in  $\Gamma'$ . By contrast, the maximin strategies in  $\Gamma$  disappears. That is, the new maximin strategy in  $\Gamma'$  is ‘I’ for both players and it guarantees zero.<sup>9</sup> On the other hand, all maximin equilibria including (b,b) remains unchanged in  $\Gamma'$ .

### 3 The mixed extension of games

#### 3.1 Maximin equilibrium

The mixed extension of a two-player non-cooperative game is denoted by  $(\Delta X_1, \Delta X_2, u_1, u_2)$  where  $\Delta X_i$  is the set of all simple probability distributions over the set  $X_i$ .<sup>10</sup> It is assumed that the preferences of the players over the strategy profiles satisfy weak order, continuity and the independence axioms.<sup>11</sup> As a result, those preferences can be represented by von Neumann-Morgenstern (expected) utility functions  $u_1, u_2 : \Delta X_1 \times \Delta X_2 \rightarrow \mathbb{R}$ . A mixed strategy profile is denoted by  $p \in \Delta X$  where  $\Delta X = \Delta X_1 \times \Delta X_2$ .

We do not need another definition for maximin equilibrium with respect to mixed strategies; one can just interpret the strategies in Definition 2 and in Definition 3 as being mixed. Harsanyi and Selten (1988, p. 70) argue that invariance with respect to positive linear transformations of the utilities is a fundamental requirement for a solution concept. The following proposition

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<sup>9</sup>It is clear that whichever game we consider, it is possible to make maximin strategies disappear by this way.

<sup>10</sup>For a detailed discussion of the mixed strategy concept, see Luce and Raiffa (1957, p. 74)’s influential book in game theory.

<sup>11</sup>For more information see, for example, Fishburn (1970).

shows that maximin equilibrium has this property.

**Proposition 2.** *The maximin equilibria of a game in mixed extension is unique up to positive linear transformations of the utilities.*

We omit the proof since it follows essentially the same steps as the proof of Proposition 1. The following lemma illustrates a useful property of the value function of a player.

**Lemma 1.** *The value function of a player is upper semi-continuous.*

*Proof.* In several steps, we show that the value function  $v_i$  of player  $i$  in a game  $\Gamma = (\Delta X_1, \Delta X_2, u_1, u_2)$  is upper semi-continuous.

Firstly, we show that the better reply correspondence  $B_j : \Delta X_i \times \Delta X_j \rightarrow \Delta X_j$  is lower hemi-continuous. For this, it is enough to show the graph of  $B_j$  defined as follows is open.

$$Gr(B_j) = \{(q, p_j) \in \Delta X \times \Delta X_j \mid p_j \in B_j(q)\}.$$

$Gr(B_j)$  is open in  $\Delta X \times \Delta X_j$  if and only if its complement is closed. Let  $[(p_j, q_i, q_j)]_{k=1}^\infty$  be a sequence in  $[Gr(B_j)]^c = (\Delta X \times \Delta X_j) \setminus Gr(B_j)$  converging to  $(p_j, q_i, q_j)$  where  $p_j^k \notin B_j(q^k)$  for all  $k$ . That is, we have  $u_j(p_j^k, q_i^k) \leq u_j(q^k)$  for all  $k$ . Continuity of  $u_j$  implies that  $u_j(p_j, q_i) \leq u_j(q)$  which means  $p_j \notin B_j(q)$ . Hence  $[Gr(B_j)]^c$  is closed which implies  $B_j$  is lower hemi-continuous.

Next, we define  $\hat{u}_i : \Delta X_i \times \Delta X_j \times \Delta X_j \rightarrow \mathbb{R}$  by  $\hat{u}_i(q_i, q_j, p_j) = u_i(p_j, q_i)$  for all  $(q_i, q_j, p_j) \in \Delta X_i \times \Delta X_j \times \Delta X_j$ . Since  $u_i$  is continuous,  $\hat{u}_i$  is also continuous. In addition, we define  $\bar{u}_i : Gr(B_j) \rightarrow \mathbb{R}$  as the restriction of  $\hat{u}_i$  to  $Gr(B_j)$ , i.e.  $\bar{u}_i = \hat{u}_i|_{Gr(B_j)}$ . The continuity of  $\hat{u}_i$  implies the continuity of its restriction  $\bar{u}_i$  which in turn implies  $\bar{u}_i$  is upper semi-continuous.

By the theorem of Berge (1959, p. 115)<sup>12</sup> lower hemi-continuity of  $B_j$  and lower semi-continuity of  $-\bar{u}_i : Gr(B_j) \rightarrow \mathbb{R}$  implies that the function  $-\bar{v}_i : \Delta X_i \times \Delta X_j \rightarrow \mathbb{R}$  defined by  $-\bar{v}_i(q) = \sup_{p_j \in B_j(q)} -\bar{u}_i(p_j, q)$  is lower semi-continuous.<sup>13</sup> It implies that the function  $\bar{v}_i(q) = \inf_{p_j \in B_j(q)} \bar{u}_i(p_j, q)$  is upper semi-continuous.

As a result, the value function of player  $i$  defined by  $v_i(q) = \min\{\bar{v}_i(q), u_i(q)\}$  is upper semi-continuous because the minimum of two upper semi-continuous functions is also upper semi-continuous.  $\square$

<sup>12</sup>We follow the terminology, especially the definition of upper hemi-continuity, presented in Aliprantis and Border (1994, p. 569).

<sup>13</sup>We use the fact that a function  $f$  is lower semi-continuous if and only if  $-f$  is upper semi-continuous.

	A	B	C	D
A	2, 2	0, 0	1, 1	0, 0
B	0, 0	90, 80	3, 3	90, 90
C	1, 100	100, 80	-1, -1	-3, -2
D	3, 1	75, 0	0, 0	230, 0

Figure 4: A game  $\Gamma$  in mixed extension.

The following theorem shows that maximin equilibrium exists also in mixed strategies.

**Theorem 2.** *Every finite game in mixed extension has a maximin equilibrium.*

*Proof.* Let us define  $v_i^{max} = \arg \max_{q \in \Delta X} v_i(q)$  which is a non-empty compact set because  $\Delta X$  is compact and  $v_i$  is upper semi-continuous by Lemma 1. Since  $v_i^{max}$  is compact and  $v_j$  is also upper semi-continuous the set  $v_{ij}^{max} = \arg \max_{q \in v_i^{max}} v_j(q)$  is non-empty and compact. Clearly, the profiles in  $v_{ij}^{max}$  are Pareto optimal with respect to the value function which means  $v_{ij}^{max}$  is a non-empty compact subset of the set of maximin equilibria in the game. Similarly, one may show that the set  $v_{ji}^{max}$  is also a non-empty compact subset of the set of the maximin equilibria.  $\square$

For an illustrative example, let us consider the game in Figure 4 played by Alfa and Beta. Observe that it has a unique Nash equilibrium (D,A) whose payoff vector is (3,1). An interesting phenomenon occurs if we change, ceteris paribus, the payoff of  $u_1(C, D)$  from  $-3$  to  $-4$ . Let us call the new game  $\Gamma'$ . It has the same pure Nash equilibrium (D,A) as  $\Gamma$  plus two mixed ones. Among them, the Pareto dominant Nash equilibrium is  $[(0, \frac{41}{46}, \frac{5}{46}, 0), (0, \frac{47}{52}, 0, \frac{5}{52})]$  whose expected payoff vector is (90, 80).<sup>14</sup> Note that by passing from  $\Gamma'$  to  $\Gamma$  we just slightly increase Alfa's relative preference of the worst outcome (C,D) with respect to the other outcomes and also that ordinal preferences remain the same. From economics viewpoint the question arises: Should ceteris paribus effect of increasing the payoff of  $u_1(C, D)$  from  $-4$  to  $-3$  be substantially high with respect to the solutions of the two games? According to maximin equilibrium the answer is negative. For instance, there is a

<sup>14</sup>The other Nash equilibrium is approximately  $[(0, 0.01, 0.001, 0.98), (0.20, 0.88, 0, 0.09)]$  whose expected payoff vector is approximately (88.11, 1.14).



	F	O		F	O
F	2, 1	0, 0	F	2, 2	0, 1
O	0, 0	1, 2	O	0, 1	1, 3

Figure 5: Two strategically equivalent battle of the sexes games.

maximin equilibrium  $[B, (0, \frac{28}{31}, 0, \frac{3}{31})]$  in  $\Gamma$  whose value is approximately 80.9 for both players. Moreover, it remains to be a maximin equilibrium with the same value in  $\Gamma'$ .<sup>15</sup> Actually, it turns out that the value of a player at a strategy profile is continuous as a function of her utility at this profile. The following proposition shows this result formally.

**Proposition 3.** *Let  $\Gamma = (\Delta X_1, \Delta X_2, u_1, u_2)$  be a game and fix a strategy profile  $p \in \Delta X_1 \times \Delta X_2$ . Everything else being equal, if we increase (decrease)  $u_i(p)$  by  $\epsilon > 0$  then  $v_i(p)$  weakly increases (decreases) by at most  $\epsilon$ .*

*Proof.* There are two cases. Case 1: Define  $\inf_{p_2 \in B_2(p)} u_1(p_1, p_2) = \underline{u}_1$  and suppose that  $u_1(p) > \underline{u}_1$  so that  $v_1(p) = \underline{u}_1$ . Then, for the new value  $v'_1$  we still have  $v'_1(p) = \underline{u}_1$  so  $v_1(p)$  remains unchanged. Case 2: Suppose that  $u_1(p) \leq \underline{u}_1$  so that  $u_1(p) = v_1(p)$ . If  $\underline{u}_1 < u_1(p) + \epsilon$  then we have  $v'_1(p) = \underline{u}_1 < u_1(p) + \epsilon = v_1(p) + \epsilon$ . If  $\underline{u}_1 \geq u_1(p) + \epsilon$  then  $v'_1(p) = u_1(p) + \epsilon = v_1(p) + \epsilon$ . The case when the value of a player decreases can be shown by following similar steps as above.  $\square$

Since the above proposition is true for every profile, it also holds for maximin equilibria. Note also that increasing the utility of a player at a profile does not affect the value of the player at the other profiles. Hence, suppose we increase Alfa's payoff of any profile by  $\epsilon > 0$  in a game  $\Gamma$  and call the new game  $\Gamma'$ . Then it is not possible to find a maximin equilibrium  $p$  in  $\Gamma$  so that Alfa's value at  $p$  is strictly larger than Alfa's value of any maximin equilibrium in  $\Gamma'$ .

For another illustrative example consider the battle of the sexes game presented on the left in Figure 5. Alfa and Beta have each two choices to make between 'Opera' (O) and 'Football' (F). There are two maximin equilibria in

<sup>15</sup>Note that we have given one example of maximin equilibrium whose value is equal for both players, but there can be other maximin equilibria as well. In addition, the maximin equilibrium is given with respect to the mixed extension of the game. If we do not allow for mixed strategies, then (B,B) would be the only maximin equilibrium in deterministic games  $\Gamma$  and  $\Gamma'$ .

this game which are (O,O) and (F,F) that are also Nash equilibria. Given the information, it does not seem possible to define a unique ‘solution’ to this game. One might be tempted to propose that the solution of this game should be the mixed Nash equilibrium  $[(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})]$  whose expected payoff vector is  $(\frac{2}{3}, \frac{2}{3})$  because it seems more ‘distinguishable’. This temptation, however, may disappear when we consider the game on the right in Figure 5. In this game, it seems that the profile (F,F) is also ‘distinguishable’ and it Pareto dominates the mixed Nash equilibrium  $[(\frac{2}{3}, \frac{1}{3}), (\frac{1}{3}, \frac{2}{3})]$  whose payoff is  $(\frac{2}{3}, \frac{5}{3})$ . Notice that the payoffs of Beta in the second game is just a positive linear transformation of the payoffs in the first game. Therefore, these two games must have the same solution in whatever way we define it; assuming that a solution must be invariant with respect to different numerical representation of the utilities.

### 3.2 The relation of maximin equilibrium with the other concepts

Nash equilibrium is probably the most well-known solution concept in game theory. Let us state Nash (1950)’s path-breaking theorem formally: Every finite game in mixed extension possesses at least one strategy profile  $p$  such that  $p_i \in \arg \max_{p'_i \in \Delta X_i} u_i(p'_i, p_j)$ . The following two propositions illustrate Pareto dominance relation between Nash equilibrium and maximin equilibrium.

**Proposition 4.** *For every Nash equilibrium that is not a maximin equilibrium there exists a maximin equilibrium that Pareto dominates it.*

*Proof.* If a Nash equilibrium  $q$  in a game is not a maximin equilibrium, then there exists a maximin equilibrium  $p$  whose value  $v(p)$  Pareto dominates  $v(q)$ . It implies that  $p$  Pareto dominates  $q$  in the game since the payoff vector of the Nash equilibrium  $q$  is the same as its value.  $\square$

The following corollary shows that a strong Nash equilibrium (Aumann, 1959) is always a maximin equilibrium.

**Corollary 1.** *A strong Nash equilibrium is a maximin equilibrium.*

*Proof.* Suppose that a profile is a strong Nash equilibrium. Then it is Pareto optimal and there is no individually rational deviation from it which implies that it is a maximin equilibrium.  $\square$

**Proposition 5.** *A maximin equilibrium is never Pareto dominated by a Nash equilibrium.*

*Proof.* By contradiction, suppose that a Nash equilibrium  $q$  Pareto dominates a maximin equilibrium  $p$ . It implies that the value of  $q$  also Pareto dominates the value of  $p$ . But this is a contradiction to our supposition that  $p$  is a maximin equilibrium.  $\square$

The two propositions above are closely linked but one does not follow from the other. Proposition 4 does not exclude the existence of a Nash equilibrium that is both Pareto dominated by a maximin equilibrium and Pareto dominates another maximin equilibrium. Proposition 5 shows that this is not the case.

Note that maximin equilibrium is distinct from rationalizable strategy profiles (Bernheim, 1984 and Pearce, 1984) and correlated equilibrium (Aumann, 1974) since maximin equilibrium is not necessarily an outcome of the iterated elimination of strictly dominated strategies. As discussed earlier, the profile (2,2) is the only outcome of this process in the traveler's dilemma, but it is not a maximin equilibrium.

One might wonder whether there is a relationship between the maximin (minimax) decision rule<sup>16</sup> in decision theory and the maximin equilibrium. Imagine a one-player game in which the decision maker is to make a choice between several gambles. In that case, maximin equilibrium boils down to expected utility maximization just like maximin strategies and Nash equilibrium. In other words, the decision maker has to choose the gamble with the highest expected utility. However, according to maximin decision rule, a decision maker has to choose the gamble which maximizes the utility with respect to the worst state of the world (whose outcome is the minimum) even though the probability assigned to it is very small.

## 4 Zerosum games

Two-person zerosum games are both a historically and theoretically important class in game theory. We illustrate the relationship between the equilibrium solution of von Neumann (1928) and the maximin equilibrium in this class of games. The following lemma will be useful for the next proposition.

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<sup>16</sup>See Wald (1950) for maximin decision rule and see Gilboa and Schmeidler (1989) for an axiomatization of it.

**Lemma 2.** *Let  $(Y_1, Y_2, u_1, u_2)$  be a two-person zerosum game where  $Y_i$  is not necessarily finite. Then  $v_i(y_i, y_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j)$  for each  $i \neq j$ .*

*Proof.* Suppose that there exists  $\bar{y}_j \in Y_j$  such that  $\bar{y}_j \in \arg \min_{y'_j \in Y_j} u_i(y_i, y'_j)$ . Then  $v_i(y_i, y_j) = \min_{y'_j \in Y_j} u_i(y_i, y'_j) = u_i(y_i, \bar{y}_j)$ . Suppose, otherwise, that for all  $y'_j \in Y_j$  there exists  $y''_j \in Y_j$  such that  $u_i(y_i, y''_j) < u_i(y_i, y'_j)$ . It implies that  $v_i(y_i, y_j) = \inf_{y'_j: u_i(y_i, y'_j) < u_i(y_i, y_j)} u_i(y_i, y'_j) = \inf_{y'_j \in Y_j} u_i(y_i, y'_j)$ .  $\square$

The following proposition shows that a strategy profile is a maximin equilibrium if and only if it is a pair of maximin strategies in zerosum games.

**Proposition 6.** *Let  $(Y_1, Y_2, u_1, u_2)$  be a two-person zerosum game where  $Y_i$  is not necessarily finite. A profile  $(y_1^*, y_2^*) \in Y_1 \times Y_2$  is a maximin equilibrium if and only if  $y_1^* \in \arg \max_{y_1} \inf_{y_2} u_1(y_1, y_2)$  and  $y_2^* \in \arg \max_{y_2} \inf_{y_1} u_2(y_1, y_2)$ .*

*Proof.* ‘ $\Rightarrow$ ’ Firstly, we show that the value of a maximin equilibrium  $(y_1^*, y_2^*)$  must be Pareto dominant in a zerosum game. By contraposition, suppose that its value is not Pareto dominant, i.e. there is another maximin equilibrium  $(\hat{y}_1, \hat{y}_2)$  such that  $v_i(y_1^*, y_2^*) > v_i(\hat{y}_1, \hat{y}_2)$  and  $v_j(y_1^*, y_2^*) < v_j(\hat{y}_1, \hat{y}_2)$  for  $i \neq j$ . By Lemma 2, we have  $v_1(y_1^*, y_2^*) = v_1(y_1^*, \hat{y}_2)$  and  $v_2(\hat{y}_1, \hat{y}_2) = v_2(y_1^*, \hat{y}_2)$ . It implies that the value of  $(y_1^*, \hat{y}_2)$  Pareto dominates the value of  $(y_1^*, y_2^*)$  which is a contradiction to our supposition that  $(y_1^*, y_2^*)$  is a maximin equilibrium. Since the value of  $(y_1^*, y_2^*)$  is Pareto dominant, each strategy is a maximin strategy of the respective players.

‘ $\Leftarrow$ ’ Suppose that for each  $i$  we have  $y_i^* \in \arg \max_{y_i} \inf_{y_j} u_i(y_i, y_j)$ . By Lemma 2, it implies that for all  $(y'_1, y'_2) \in Y_1 \times Y_2$  and for each  $i$  we have  $v_i(y_1^*, y_2^*) \geq v_i(y'_1, y'_2)$ . Hence the value of  $(y_1^*, y_2^*)$  is Pareto dominant which implies that it is a maximin equilibrium.  $\square$

**Corollary 2.** *In a zerosum game, maximin equilibrium and equilibrium coincide whenever an equilibrium exists.*

As a result, maximin equilibrium indeed generalizes the maximin strategy concept of von Neumann (1928) from zerosum games to non-zerosum games. To sum up, existence of an equilibrium in a zerosum game implies that equilibria and maximin equilibria coincide. But note that maximin equilibrium may exist even though an equilibrium does not exist. In any case, maximin equilibrium is a pair of maximin strategies in zerosum games.

For an illustrative example let us consider the following game to be played by Alfa and Beta at a television program. Initially, Beta has to make a choice

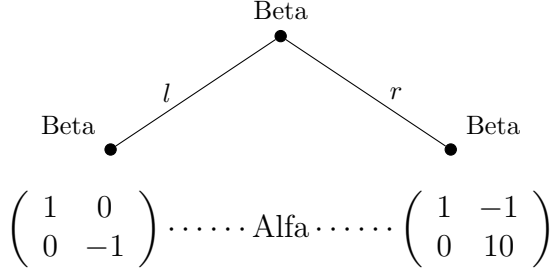


Figure 6: The game  $(\Delta X, \Delta X_l \cup \Delta X_r, u, -u)$ .

between the left door and the right door. She is not allowed to commit to a randomization device nor is she allowed to use a device by herself for this choice. If she picks the left door, they will play the game at the left of Figure 6. If she picks the right door, they will play the game at the right of Figure 6. At this stage, players may commit to mixed strategies by submitting them on a computer. Alfa will not be informed which normal-form game he is playing. This situation can be represented by the zerosum game  $(\Delta X, \Delta X_l \cup \Delta X_r, u, -u)$  in which Alfa chooses a mixed strategy in  $\Delta X$  and Beta chooses a mixed strategy in either  $\Delta X_l$  or in  $\Delta X_r$ .

Notice that there is no equilibrium in this game. There are, however, maximin strategies for each player that are  $(\frac{11}{12}, \frac{1}{12}) \in \Delta X$  guaranteeing  $\frac{-1}{12}$  and  $(0, 1) \in \Delta X_l$  guaranteeing 0. By Proposition 6, this pair is also the unique maximin equilibrium whose payoff vector is  $(\frac{-1}{12}, \frac{1}{12})$ . However, maximin equilibrium does not necessarily say that this is the payoff that players should expect by playing their part of the maximin equilibrium. Rather, the unique maximin equilibrium value of this game is  $(\frac{-1}{12}, 0)$ . In other words, the unique value of the game to Alfa is  $\frac{-1}{12}$  given the individual rationality of Beta and the unique value of the game to Beta is 0 given the individual rationality of Alfa. If the television programmer modifies the game so that Beta is allowed to commit to a randomization device in the beginning, then the game would have an equilibrium  $[(\frac{11}{12}, \frac{1}{12}), (0, \frac{11}{12}, 0, \frac{1}{12})]$  which is also a maximin equilibrium. Note that Beta is now able to guarantee the payoff  $\frac{1}{12}$ . As a result, the unique value of the modified game would be  $(\frac{-1}{12}, \frac{1}{12})$ .

Speaking of the importance of committing to mixed strategies, let us consider the following zerosum game in Figure 7 which was discussed in Aumann and Maschler (1972). Suppose that players cannot commit playing mixed strategies but a randomization device, e.g. a coin, is available. Be-

	L	R
L	0, 0	2, -2
R	3, -3	1, -1

Figure 7: A zerosum game.

fore the coin toss, the maximin strategy  $(\frac{1}{2}, \frac{1}{2})$  of Alfa guarantees the highest expected payoff of 1.5 in the mixed extension. However, after the coin toss Alfa still needs to make a decision whether playing according to the outcome of the toss or not. Actually, for both players playing strategy R guarantees more than playing L after the randomization. Hence the maximin equilibrium of this deterministic game is (R,R) whose value is  $(1, -2)$  whereas the values of the profiles (L,L), (L,R) and (R,L) are  $(0, -3)$ ,  $(0, -2)$  and  $(1, -3)$  respectively. Note that if players are allowed to use mixed strategies then the maximin equilibrium is  $[(\frac{1}{2}, \frac{1}{2}), (\frac{1}{4}, \frac{3}{4})]$ .

## 5 Maximin equilibrium in $n$ -person games

Firstly, we define the value function. For this, we replace the way  $v_i$  is written in Definition 2 to  $v_i(p) = \min\{\inf_{p'_{-i} \in B_{-i}(p)} u_i(p_i, p'_{-i}), u_i(p)\}$  where  $B_{-i}(p)$  is defined as follows. Firstly for each  $S \subseteq N \setminus \{i\}$  and each  $p \in \Delta X$  define

$$B_{-i}^S(p) = \{(\hat{p}_S, p_{-S}) \in \Delta X_{-i} \mid u_k(\hat{p}_k, p_{-k}) > u_k(p) \text{ for all } k \in S\}.$$

$B_{-i}^S(p)$  is the set of  $(n-1)$ -tuple strategy profiles in which the players in  $S$  make a *unilateral* profitable deviation with respect to  $p$ . To represent all such profiles for all  $S \subseteq N \setminus \{i\}$ , we define the correspondence  $B_{-i}(p) = \bigcup_{S \subseteq N \setminus \{i\}} B_{-i}^S(p)$ . Accordingly, a strategy profile is a maximin equilibrium if its value is not Pareto dominated. Moreover, every result in Section 2 and in Section 3 is valid in  $n$ -person games. The proofs are essentially the same as the ones given in Section 2 and in Section 3.

Even in a purely non-cooperative framework, strategic thinking in  $n$ -person games may be quite different than in two-person games. Let us consider the game in Figure 8 played by Alfa, Beta and Juliet to show that even the unique Nash equilibrium can be “fragile” in games with more than two players. This game has a unique Nash equilibrium which is approximately  $[(0.65, 0.35, 0), (0.25, 0.75), (0.68, 0.32)]$  whose payoff vector is approximately

	D	E		D	E
A	1, 1, 1	0, 0, 1	A	-2, -1, 6	3, 3, -2
B	4, 6, 2	0, 4, 6	B	3, 4, 3	-1, 5, 8
C	-2, 1, 1	0, 0, 2	C	4, 0, 0	-1, 0, 1

Figure 8: A three player game where player 3 chooses between the matrices L (left) and R (right).

(0.71, 2.12, 2.39). Note that the Nash-value of Juliet is the highest so she seems to be the most advantageous player in the game. Suppose that Juliet naively thinks that she is doing the ‘best’ by playing her part of the Nash equilibrium. Even without any communication, Alfa and Beta may unilaterally deviate from the Nash equilibrium to the strategies ‘B’ and ‘C’ respectively after which they both receive (3.68 and 5.36, respectively) strictly more than their Nash equilibrium payoff which causes the Nash equilibrium to “break down”. As a result, Juliet ends up with a strictly less payoff (2.32) than her payoff at the Nash equilibrium.

Notice that potential deviations of Alfa and Beta are costless, because the strategy ‘B’ of Alfa is a best response to the Nash equilibrium strategies of the other players and ‘D’ of Beta is also best response to the Nash equilibrium strategies of the others. Note also that these deviations are not coalitional deviations. We do not claim that when a player deviates, the other also must deviate. It could very well be the case that Alfa unilaterally deviates to ‘B’ but Beta sticks to her Nash equilibrium strategy or vice versa. In this case, Alfa would not lose anything. What ‘breaks the Nash equilibrium down’ is the very possibility that by anticipating the situation Beta also deviates to ‘D’. In addition, holding the Nash equilibrium strategy (0.68, 0.32) of Juliet fixed, the profile (B,D) is the Pareto dominant Nash equilibrium in the game played by Alfa and Beta! Consequently, the very argument that players have no incentive to unilaterally deviate at a Nash equilibrium does not hold in this example. Since every pure strategy in the support of a mixed Nash equilibrium is a best response, every mixed Nash equilibrium and even sometimes a pure Nash equilibrium may, potentially, have the problem described above in  $n$ -person games.<sup>17</sup>

<sup>17</sup>It is well-known that a Nash equilibrium is not necessarily immune to profitable coalitional deviations. Therefore some refinements of Nash equilibrium has been proposed such as strong Nash equilibrium (Aumann, 1959) and coalition-proof Nash equilibrium (Bern-

In fact, von Neumann and Morgenstern (1944, p. 32) strikingly anticipate the problem we discussed above years before the emergence of Nash equilibrium: “Imagine that we have discovered a set of rules for all participants to be termed as “optimal” or “rational” each of which is indeed optimal provided that the other participants conform. Then the question remains as to what will happen if some of the participants do not conform. If that should turn out to be advantageous for them and, quite particularly, disadvantageous to the conformists then the above “solution” would seem very questionable. We are in no position to give a positive discussion of these things as yet but we want to make it clear that under such conditions the “solution,” or at least its motivation, must be considered as imperfect and incomplete.”

Maximin equilibrium can be modified to incorporate coalitions in  $n$ -person games, we just need to define the better reply correspondence allowing coalitional profitable deviations and define the value function with respect to this. Accordingly, a profile is called strong maximin equilibrium if its value is not Pareto dominated. By the same argument in Theorem 1, it exists in pure strategies in the deterministic game. Regarding the mixed extension of games, one may show the existence of strong maximin equilibrium by following the similar steps as in Lemma 1 and in Theorem 2. Regarding the three-player game above, both the maximin equilibrium and the strong maximin equilibrium is the profile  $(B, D, (\frac{1}{2}, \frac{1}{2}))$  whose value is  $(3, 4, 2.5)$ . In other words, by playing their part of the maximin equilibrium each player guarantees her value under any profitable deviation of the other players.

## 6 Conclusion

In this paper, we extended von Neumann’s maximin strategy solution in strategic games by incorporating individual rationality of the players. Maximin equilibrium extends Nash’s value approach to the whole game and evaluates the strategic uncertainty of the game by following a similar method as von Neumann’s maximin strategy notion. We showed that maximin equilibrium is invariant under strictly increasing transformations of the payoffs. Notably, every finite game possesses a maximin equilibrium in pure strategies.

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heim et al., 1987). These concepts, however, have the non-existence problem and they are sometimes interpreted with pre-play communication.



Considering the games in von Neumann-Morgenstern mixed extension, we demonstrated that maximin equilibrium exists in mixed strategies as well. Moreover, we showed that a strategy profile is a maximin equilibrium if and only if it is a pair of maximin strategies in two-person zerosum games. In particular, a maximin equilibrium and an equilibrium coincide whenever the latter exist in those games. Furthermore, we showed that for every Nash equilibrium that is not a maximin equilibrium there exists a maximin equilibrium that Pareto dominates it. Hence, a strong Nash equilibrium is always a maximin equilibrium. Besides, a maximin equilibrium is never Pareto dominated by a Nash equilibrium. The concept introduced in this paper opens up several research directions such as further exploration in extensive form games and in repeated games.

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